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Nonexistence of codimension one Anosov flows on compact manifolds with boundary ☆

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Abstract

A flow is *Anosov* if it exhibits contracting and expanding directions forming with the flow a continuous tangent bundle decomposition. An Anosov flow is *codimension one* if its contracting or expanding direction is one-dimensional. Examples of codimension one Anosov flows on compact boundaryless manifolds can be exhibited in any dimension $\geqslant 3$. In this paper, we prove that there are no codimension one Anosov flows on compact manifolds *with boundary*. The proof uses an extension to flows of some results in Hirsch [On Invariant Subsets of Hyperbolic Sets, Essays on Topology and Related Topics, Memoires dédiés à Georges de Rham, 1970, pp. 126–135] related to Question 10(b) in Palis and Pugh [Fifty problems in dynamical systems, in: J. Palis, C.C. Pugh (Eds.), Dynamical Systems-Warwick 1974 (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E.C. Zeeman on his fiftieth birthday), Lecture Notes in Mathematics, vol. 468, Springer, Berlin, 1975, pp. 345–353].

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1. Introduction

Dynamical systems on manifolds with boundary have been studied elsewhere in the literature. For instance, [20] considered the open-denseness of structural stable systems among the transient vector fields on a manifold with boundary. In [2,3,22] the classical characterization of structural and codimension one stable flows on closed surfaces was extended to the boundary case. In [21] the author gives sufficient conditions for a C^1 flow ϕ on a compact manifold with boundary M to be weakly structurally stable. No restrictions are imposed on the tangencies of ϕ to the boundary. In [15,17] the theory of Morse–Smale systems on closed manifolds was extended to the boundary case. Indeed, they considered an open-dense subset $\mathcal{X}_*^\infty(M, \partial M)$ of the space $\mathcal{X}^\infty(M, \partial M)$ of C^∞ vector fields tangent to ∂M, where M is a compact manifold with boundary ∂M. The authors define the Morse–Smale vector fields in order to prove the equivalence between structural stability and Morse–Smale among the elements of $\mathcal{X}_*^\infty(M, \partial M)$ with simple non-wandering set. In [14] the authors construct a vector field on the three-dimensional unit disc tangent to the boundary which is C^1 structurally stable but not hyperbolic. This differs from the boundaryless case, where C^1 structural stability implies the hyperbolicity of the non-wandering set [9]. The study of dynamical systems on manifolds with boundary is related to the study of equivariant dynamical systems [4,5].

In this paper, we prove that there are no codimension one Anosov flows on compact manifolds with boundary. A related work is [10], where it is proved the non-existence of positively expanding maps on compact manifolds with boundary. See also [6, Section 5, p. 575], where examples of $\mathbb{Z}_2$-Anosov diffeomorphisms in the two-torus T^2 are exhibited ($\mathbb{Z}_2$ is the cyclic group of order 2). Besides the conclusion of Theorem 1 is contrary to the boundaryless case where examples of codimension one Anosov flows can be exhibited in any dimension $\geqslant 3$. The proof of Theorem 1 uses an extension to flows of some results in [11] related to Question 10(b) in [18].

Let us state the result in a precise way. Hereafter, M is a compact manifold and ∂M denotes the boundary of M. We say that M is a *manifold with boundary* if $\partial M \neq \emptyset$. See [12,16] for references concerning manifolds with boundary. A C^r flow in M is a C^r-action $\phi : I\!R \times M \to M$ of the additive group $I\!R$ on M, $r \geqslant 1$. We denote by ϕ_t the time t-map $\phi_t(x) = \phi(t, x)$ of ϕ. We say that $\Lambda \subset M$ is *ϕ-invariant* if $\phi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. It is clear by invariance of domain that the boundary ∂M is a codimension one invariant submanifold of ϕ.

Definition 1. Let ϕ be a C^1-flow on a manifold M possibly with boundary. A compact ϕ-invariant set $\Lambda \subset M$ is *hyperbolic* if there is a continuous, invariant, tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^o_\Lambda \oplus E^u_\Lambda$ over Λ such that the following hold for some constants $C, \lambda > 0$:

1. E^s_Λ is *contracting*, namely

$$|| \, D\phi_t(x) v^s_x \, || \leqslant C e^{-\lambda t} \, || \, v^s_x \, || \quad \text{for } v^s_x \in E^s_x \text{ and } x \in \Lambda.$$

2. E^u_Λ is *expanding*, namely

$$\| D\phi_t(x)v^u_x \| \geqslant C^{-1}e^{\lambda t} \| v^u_x \| \quad \text{for } v^u_x \in E^u_x \text{ and } x \in \Lambda.$$

3. E^o_Λ is the flow direction, namely E^o_x is tangent to the curve $\{\phi_t(x) : t \in \mathbb{R}\}$ for all $x \in \Lambda$.

The decomposition $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda \oplus E^o_\Lambda$ is called *the hyperbolic splitting* of Λ.

The dimension of either a linear space or a manifold L will be denoted by $dim(L)$. The definition of Anosov flow below is related with the definition of G-Anosov flow in [6, p. 574, Section 4] with $G = \mathbb{Z}_2$.

Definition 2. A C^1 flow on a manifold M is *Anosov* if M is a hyperbolic set of it. An Anosov flow is *codimension one* if its hyperbolic splitting $TM = E^s_M \oplus E^o_M \oplus E^u_M$ satisfies either $dim(E^s_x) = 1$ for all $x \in M$ or $dim(E^u_x) = 1$ for all $x \in M$.

In this paper, we shall prove the following.

Theorem 1. *There are no codimension one Anosov flows on compact manifolds with boundary*.

The discrete version of Theorem 1 also holds, namely *there are no codimension one Anosov diffeomorphisms on compact manifolds with boundary*.

The proof of Theorem 1 goes as follows. First, we state some preliminary lemmas and observe that there are no transitive Anosov flows on compact manifolds with boundary (Corollary 1). In Proposition 1, we prove that if ϕ is a C^1 flow on a compact manifold, N is a closed submanifold in M with $dim(N) \geqslant 2$ and N is a hyperbolic set of ϕ with $dim(E^s) = 1$ everywhere in N, then ϕ_t/N is Anosov. It follows from this proposition that there are no C^1 flows on compact manifolds exhibiting a closed surface as a hyperbolic set (Corollary 2). In Lemma 4, we observe that Verjovsky's Theorem [1,23] holds on compact manifolds with boundary. The proof of Theorem 1 in dimension 3 follows from Corollary 2. The proof in dimension $\geqslant 4$ follows from Corollary 1 and Lemma 4 (this argument does not work in dimension 3 since Verjovsky's Theorem is false in that dimension [7]).

2. Proof

We start with some useful definitions. Let M be a compact manifold. A *closed submanifold in M* is a compact connected boundaryless manifold N embedded in M. If $dim(N) = 2$ we say that N is a closed surface in M. Let ϕ be a flow on M. A singularity of ϕ is a point p such that $\phi_t(p) = p$ for all t. A periodic point of ϕ is a point p such that $\phi_T(p) = p$ for some minimal $T > 0$. The full orbit of a periodic point is a periodic orbit of ϕ. We say that ϕ is non-singular if it has no singularities. We say that ϕ is *transitive* if it has a dense orbit.

The *omega-limit set* of a point x (with respect to ϕ) is the set

$$\omega_\phi(x) = \left\{ y = \lim_{n\to\infty} \phi_{t_n}(x) \text{ for some sequence } n \to \infty \right\}.$$

The *alpha-limit set* of x is the set $\alpha_\phi(x) = \omega_{-\phi}(x)$, where $-\phi$ is the reversed flow ϕ_{-t}. A compact invariant set Ω of ϕ is *transitive* if $\Omega = \omega_\phi(x)$ for some $x \in \Omega$. A point $x \in M$ is *recurrent* (for ϕ) whenever $x \in \omega_\phi(x)$. A direct consequence of the Zorn's Lemma is that every omega-limit set on a compact manifold contains a recurrent point.

If C is a compact invariant set of a flow ϕ we denote by $Per_\phi(C)$ the set of periodic points of ϕ contained in C. On the other hand, we say that C is *isolated* if there is a neighborhood U of it such that $C = \cap_{t\in\mathbb{R}} \phi_t(U)$. The closure of B is denoted by $Cl(B)$.

Now, let H be a hyperbolic set of a C^r flow ϕ on a compact manifold M. In the boundaryless case ($\partial M = \emptyset$), the Stable Manifold Theory [13] says that for every $x \in H$ the sets

$$W^{ss}(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^{uu}(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to -\infty\}$$

are C^r boundaryless submanifolds of M. These manifolds are called, respectively, the *strong stable* and *strong unstable manifolds* of x. One knows that $W^{ss}(x)$ and $W^{uu}(x)$ are tangent to the subspaces E^s_x and E^u_x of the hyperbolic splitting of H at x, respectively. One also knows that the set-valued maps $x \in H \to W^\sigma(x)$ for $\sigma = ss, uu$ are continuous in compact parts.

Remark 1. Similar fact holds in the boundary case. Indeed, let H be a hyperbolic set of a C^1 flow ϕ on a compact manifold with boundary M. We can assume that ϕ is defined on a *closed* manifold M' and that M is a codimension 0 submanifold M' (see [12, p. 151]). Applying the Stable Manifold Theory in the boundaryless case we have that for every $x \in H$ the sets $W^{ss}(x), W^{uu}(x)$ are also C^r submanifolds of M'.

It is clear that the manifolds above may intersect $M' \setminus M$. The following describes how a hyperbolic set H accumulates on the boundary of M.

Lemma 1. *Let H be a hyperbolic set of a C^1 flow ϕ on a compact manifold with boundary M. Then, the following properties hold*:

1. *If $x \in H$ is sufficiently close to ∂M, then either $\omega_\phi(x) \subset \partial M$ or $\alpha_\phi(x) \subset \partial M$.*
2. *$Cl(Per_\phi(H)) \setminus \partial M$ is closed in M.*
3. *If H is transitive, isolated and $H \cap \partial M \neq \emptyset$, then $H \subset \partial M$.*

Proof. To prove (1) we suppose by contradiction that there is a sequence $x_n \in H$ converging to some $z \in \partial M$ such that $\omega_\phi(x_n) \not\subset \partial M$ and $\alpha_\phi(x_n) \not\subset \partial M$. Obviously, $z \in H$ since H is closed. Let $T_H M = E^s_H \oplus E^u_H \oplus E^o_H$ be the hyperbolic splitting of H. We claim that $E^s_z \oplus E^u_z \subset T_z(\partial M)$. Indeed, suppose for a while that $E^s_z \not\subset T_z(\partial M)$. Then $W^{ss}(z) \pitchfork \partial M$ at z. By the continuity of the stable manifolds of ϕ we conclude that, for n large, $W^{ss}(x_n) \pitchfork \partial M$ at some point w. Since $w \in W^{ss}(x_n)$ we have that w and x_n have the same omega-limit sets. But the one of w is contained in ∂M since ∂M is closed invariant for ϕ. Hence, $\omega_\phi(x_n) \subset \partial M$ contrary to the assumption. This proves $E^s_z \subset T_z(\partial M)$. Analogously we prove $E^u_z \subset T_z(\partial M)$ by considering the alpha-limit set. The claim follows. The claim would imply $T_z M = E^s_z \oplus E^s_z \oplus E^o_z \subset T_z(\partial M)$ since the flow is tangent to ∂M which is absurd. This proves (1). To prove (2) we assume by contradiction that $Cl(Per_\phi(H)) \setminus \partial M$ is *not* closed in M. Then, every neighborhood of ∂M contains a periodic point $p \in H \setminus \partial M$. Since p is periodic we have $p \in \omega_\phi(p) = \alpha_\phi(p)$. It would follow from part (1) that $p \in \partial M$ which is absurd. To prove (3) we assume by contradiction that H is transitive, isolated, $H \cap \partial M \neq \emptyset$ and $H \not\subset \partial M$. Since transitive sets for flows are connected we can arrange by the Shadowing Lemma for flows [8] a periodic point sequence in $Per_\phi(H) \setminus \partial M$ converging to some point in ∂M. This contradicts (2) and the proof follows. $\square$

Corollary 1. *There are no transitive Anosov flows on compact manifolds with boundary.*

Proof. If ϕ were a transitive Anosov flow on a compact manifold with boundary M, then $H = M$ is a transitive isolated hyperbolic set of ϕ intersecting ∂M. By Lemma 1(3) we would have $M \subset \partial M$ which is absurd. $\square$

Next, we state an useful definition. A hyperbolic set H of a C^1 flow ϕ is *expanding* if its hyperbolic splitting $T_H M = E^s_H \oplus E^o_H \oplus E^u_H$ satisfies $E^s_x = 0$ and $E^u_x \neq 0$ for every $x \in H$.

Lemma 2. *Let Y be a non-singular C^1 flow on a compact manifold N with $dim(N) \geqslant 2$. If $x \in N$ has hyperbolic expanding omega-limit set, then x is a periodic point of Y.*

Proof. Since $\omega_Y(x)$ is hyperbolic expanding it follows that every point $y \in \omega_Y(x)$ has a strong unstable manifold $W^{uu}_Y(y)$ in N [13]. Since Y is non-singular one has that $dim(E^u) = dim(N) - 1$ everywhere in $\omega_Y(x)$. Hence, $dim(W^{uu}_Y(y)) = dim(N) - 1 \neq 0$ for every $y \in \omega_Y(x)$. Pick a recurrent point $y \in \omega_Y(x)$ and consider a small $(dim(N) - 1)$-dimensional disk $D \subset W^u_Y(x)$ centered in y. Denote by Π the return map induced by the reversed flow Y_{-t} in D. Since $D \subset W^u_Y(y)$ we have that Π is contracting. The Contracting Map Theorem then implies that Π has a unique fixed point which is attracting. This fixed point produces a repelling periodic orbit of Y contained in $\omega_Y(x)$. The existence of such a periodic orbit in $\omega_Y(x)$ clearly implies that the orbit of x is periodic. This proves the result. $\square$

Lemma 3. *Let ϕ be a C^1 flow on a compact manifold M. Let N be a closed sub-manifold in M which is also a hyperbolic set of ϕ with hyperbolic splitting $T_N M =$*

$E^s_N \oplus E^o_N \oplus E^u_N$. Suppose that dim$(N) \geqslant 2$. If E^s is one-dimensional (i.e. dim$(E^s_x) = 1$ for every $x \in N$), then $E^s_x \subset T_x N$ for every $x \in N$.

Proof. Denote by $Y_t = \phi_t/N$ the restricted flow. N (as every hyperbolic set of ϕ) has finitely many singularities of ϕ. Hence, the set of regular (i.e. non-singular) points is dense in N. Since N is a connected submanifold and the splitting $T_N M = E^s_N \oplus E^o_N \oplus E^u_N$ is continuous we conclude that Y is non-singular. The proof of the lemma will use the following claims.

Claim 1. *If $E^s_x \not\subset T_x N$, then $T_y N = (E^u_y \cap T_y N) \oplus E^o_y$ for every $y \in \omega_Y(x)$.*

Proof. It suffices to show that $T_y N \subset E^u_y \oplus E^o_y$ for every $y \in \omega_Y(x)$. To prove it we introduce some useful notations. For every tangent vector $Z \in T_\Lambda M$ we write

$$Z = Z^s + Z^o + Z^u$$

to indicate the components of Z in the splitting $T_\Lambda M = E^s_N \oplus E^o_N \oplus E^u_N$. Define the subbundle $E^{ou} = E^o \oplus E^u$ with corresponding decomposition $Z = Z^s + Z^{ou}$. Clearly, $Z^{ou} = Z^o + Z^u$.

Let C, λ be the constants in the definition of hyperbolicity (Definition 1). Since ϕ_t/N is non-singular (and N is compact) we have that there is a positive constant K such that

$$|| D\phi_t(x) Z^o_x || \geqslant K \, || Z^o_x || \quad \forall Z^o_x \in E^o_x.$$

As $\lambda > 0$ there is a positive constant K' such that

$$\min\{K, C^{-1} e^{\lambda t}\} \geqslant K' \quad \forall t > 0.$$

Hence, for every $x \in N$, $Z^{ou} = Z^{ou}_x \in E^{ou}_x$ and $t > 0$ we have

$$\begin{aligned} ||D\phi_t(x) Z^{ou}_x|| &= ||D\phi_t(x) Z^o_x|| + || D\phi_t(x) Z^u \,|| \\ &\geqslant K \, || Z^o_x || + C^{-1} e^{\lambda t} \, || Z^u_x || \\ &\geqslant \min\{K, C^{-1} e^{\lambda t}\}(|| Z^o_x || + || Z^u_x ||) \geqslant K' \, || Z^{ou}_x || \,. \end{aligned}$$

In conclusion, we get

$$|| D\phi_t(x) Z^{ou}_x || \geqslant K' \, || Z^{ou}_x ||, \tag{2.1}$$

for every $x \in N$, $Z^{ou}_x \in E^{ou}_x$ and $t > 0$.

Now, choose $v_y \in T_y N - 0$ and pick a sequence $t_n \to \infty$ such that $y_n := Y_{t_n}(x) \to y$ as $n \to \infty$. Since $y_n \to y$ there is another sequence $v_{y_n} \in T_{y_n} N$ such that $v_{y_n} \to v_y$ in TN as $n \to \infty$. Define

$$w_n = D\phi_{-t_n}(y_n)v_{y_n}.$$

Then $w_n \in T_x N - 0$. By normalizing w_n if necessary we can assume that $|| \, w_n \, ||= 1$ for every n.

Let us prove that there is a positive constant K'' such that

$$|| \, w_n^{ou} \, || \geqslant K'' \quad \forall n. \tag{2.2}$$

Indeed, suppose that there is no such K''. Then we can assume that $w_n^{ou} \to 0$ by passing to a subsequence if necessary. Again, by passing to a subsequence if necessary, we can assume that $w_n \to w_x$ for some vector w_x. Clearly, one has $|| \, w_x \, ||= 1$. As $w_n^{ou} \to 0$ we have $w_x \in E_x^s \cap T_x N$. But E^s is one-dimensional and $E_x^s \not\subset T_x N$ by hypothesis. Then $E_x^s \cap T_x N = 0$ from which we get $w_x = 0$, a contradiction since $|| \, w_x \, ||= 1$. We conclude that there is K'' satisfying (2.2).

Let $\angle(v_{y_n}, E_{y_n}^o \oplus E_{y_n}^u)$ be the angle between v_{y_n} and $E_{y_n}^o \oplus E_{y_n}^u$. Then we have

$$\angle(v_{y_n}, E_{y_n}^o \oplus E_{y_n}^u) = \frac{|| \, v_{y_n}^s \, ||}{|| \, v_{y_n}^{ou} \, ||}.$$

By the invariance of the hyperbolic splitting one has

$$v_{y_n}^s = D\phi_{t_n}(x)w_n^s, \quad v_{y_n}^{ou} = D\phi_{t_n}(x)w_n^{ou}.$$

Then,

$$\angle(v_{y_n}, E_{y_n}^o \oplus E_{y_n}^u) = \frac{|| \, DX_{t_n}(x)w_n^s \, ||}{|| \, DX_{t_n}(x)w_n^{ou} \, ||}.$$

From this, (2.1) and (2.2) one gets

$$\begin{aligned}\angle(v_{y_n}, E_{y_n}^o \oplus E_{y_n}^u) &\leqslant (K')^{-1}Ce^{-\lambda t_n}\frac{|| \, w_n^s \, ||}{|| \, w_n^{ou} \, ||} \\ &\leqslant (K')^{-1}(K'')^{-1}Ce^{-\lambda t_n} \, || \, w_n^s \, || \leqslant (K')^{-1}(K'')^{-1}Ce^{-\lambda t_n}\end{aligned}$$

because $|| \, w_n^s \, || \leqslant 1$ for all n (recall $|| \, w_n \, ||= 1$). Since $t_n \to \infty$ as $n \to \infty$ we get

$$\lim_{n\to\infty} \angle(v_{y_n}, E_{y_n}^o \oplus E_{y_n}^u) = 0.$$

But the continuity of the hyperbolic splitting also implies

$$\lim_{n\to\infty} \angle(v_{y_n}, E^o_{y_n} \oplus E^u_{y_n}) = \angle(v_y, E^o_y \oplus E^u_y).$$

Then

$$\angle(v_y, E^o_y \oplus E^u_y) = 0$$

which is equivalent to $v_y \in E^o_y \oplus E^u_y$. This proves $T_yN \subset E^o_y \oplus E^u_y$ and the result follows. $\square$

Claim 2. *If $x \in N$ then either $E^s_x \subset T_xN$ or $T_xN = (E^u_x \cap T_xN) \oplus E^o_x$.*

Proof. Suppose that $E^s_x \not\subset T_xN$. Then, $T_yN = (E^u_y \cap T_yN) \oplus E^o_y$ for all $y \in \omega_Y(x)$ by Claim 1. This implies that $\omega_Y(x)$ is a hyperbolic expanding set of Y. Since Y is non-singular and $dim(N) \geqslant 2$, Lemma 2 implies that x is periodic and so $x \in \omega_Y(x)$. Replacing $y = x$ above we get $T_xN = (E^u_x \cap T_xN) \oplus E^o_x$ proving the claim. $\square$

Now, we finish the proof of the lemma. Define the sets

- $B = \{x \in N : T_xN = (E^u_x \cap T_xN) \oplus E^o_x\}$;
- $C = \{x \in N : E^s_x \subset T_xN\}$.

By Claim 2 one has

$$N = B \cup C.$$

Moreover, since $E^o \subset TN$ and $T_NM = E^s_N \oplus E^u_N \oplus E^o_N$ we have that $B \cap C = \emptyset$. Let us prove that B and C are both closed in N. In fact, let $x_n \in B$ a sequence converging to $x \in N$. Since $x_n \in B$ we have $T_{x_n}N = (E^u_{x_n} \cap T_{x_n}N) \oplus E^o_{x_n}$ $\forall n$ or, equivalently, $dim(E^u_{x_n} \cap T_{x_n}N) = dim(N) - 1$ $\forall n$. By compactness, we can assume that the sequence $E^u_{x_n} \cap T x_n N$ converges to a $(dim(N)-1)$-subspace of T_xN. This subspace is necessarily contained in $E^u_x \cap T_xN$ by the continuity of E^u. Then $dim(E^u_x \oplus T_xN) = dim(N) - 1$ and so $x \in B$. This proves that B is closed. The proof for C is easier. Since $B \cap C = \emptyset$ and $N = B \cup C$ we conclude that B and C are both open and closed in N. Since $N \neq \emptyset$ one has either $B \neq \emptyset$ or $C \neq \emptyset$ and so either $N = B$ or C because N is connected. In the former case, we have that the time t mapping Y_t is volume expanding which is a contradiction. We conclude that $N = C$, i.e. $E^s_x \subset T_xN$ for every $x \in N$ and the result follows. $\square$

The following is the flow version of Theorem 7(a) [11, p. 131]. It gives a partial positive answer for the flow version of Question 10(b) in [18].

Proposition 1. *Let ϕ a C^1 flow on a compact manifold and N be a closed submanifold in M with $dim(N) \geqslant 2$. If N is a hyperbolic set of ϕ with $dim(E^s) = 1$ everywhere in N, then ϕ_t/N is Anosov.*

Proof. Let ϕ be a C^1 flow on a compact manifold M. Let N be a closed submanifold which is also a hyperbolic set of ϕ with $dim(N)\geqslant 2$. Let $T_N M = E^s_N \oplus E^o_N \oplus E^u_N$ be the hyperbolic splitting of N. Assume that $dim(E^s) = 1$. By Lemma 3, we have that $E^s \subset TN$ everywhere in N. To prove that ϕ_t/N is Anosov it suffices to prove $TN = E^s_N \oplus E^o_N \oplus (E^u_N \cap TN)$. Now, the inclusion $\supset$ is obvious since $E^s_N \oplus E^o_N \subset TN$. The inclusion $\subset$ follows by noting that if $v \in TN$ then, as in the proof of Claim 1, we can write $v = v^s + v^o + v^u$, where $v^\sigma \in E^\sigma$ $(\sigma = s, u, o)$. Hence, $v^u = v - v^o - v^s \in E^u \cap TN$ since $v^u \in E^u$ and $v, v^0, v^s \in TN$. This finishes the proof. $\square$

Corollary 2. *There are no C^1 flows on compact 3-manifolds exhibiting a closed surface as a hyperbolic set. In particular, Anosov flows on compact 3-manifolds do not have invariant closed surfaces.*

Proof. Suppose by contradiction that there is a C^1 flow ϕ on a compact 3-manifold M exhibiting a closed surface N as hyperbolic set. The contradiction will follow from Proposition 1 once we prove that $dim(E^s) = 1$ everywhere (recall that no closed surface support Anosov flows). To prove it we note that N is connected by definition. As in the proof of Claim 1 we can see that $Y_t = \phi_t/N$ is non-singular. It follows that $dim(E^o) = 1$ everywhere in N. Now, the set $\{x \in N : dim(E^s) = 2\}$ is open and closed in N by the continuity of the splitting. Analogously for $\{x \in N : dim(E^u) = 2\}$. By connectedness we conclude that $N = \{x \in N : dim(E^s) = 2\}$ or $N = \{x \in N : dim(E^u) = 2\}$. In the former case, we have that the time-t map ϕ_t/N is volume contracting (for t large) and in the latter one we have that ϕ_t/N is volume expanding (for t large). In any case we get a contradiction. This proves that $dim(E^s) = 1$ everywhere and we are done. $\square$

The following is the Verjovsky's Theorem for compact manifolds with boundary. The proof is similar to the original one (see also [1]). Recall that a *source* of ϕ is a transitive set Ω_0 of ϕ satisfying $\Omega_0 = \cap_{t\leqslant 0}\phi_t(U)$ for some neighborhood U of it. Note that if Ω_0 is a source and $p \in M$ is a hyperbolic periodic point of ϕ then $W^{ss}(x) \subset \Omega_0$. It is clear that a source is a transitive isolated set.

Lemma 4. *Let ϕ be a codimension one Anosov flow on a compact manifold with boundary M. If $dim(M)\geqslant 4$, then ϕ is transitive.*

Proof. By reversing the flow we can assume that $dim(E^u) = 1$. To apply the arguments in [23, Section 2, p. 54] (or [1, Chapitre 2]) we only has to verify the following properties:

1. There is a continuous foliation tangent to the subbundle E^u whose leaves are diffeomorphic to $\mathbb{R}$.
2. The flow ϕ has a source Ω_0 in $M \setminus \partial M$.

To verify (1) we proceed as follows. By applying Remark 1 we have that E^u is tangent to a continuous foliation in M. On the other hand, E^u is tangent to ∂M in ∂M by Lemma 3 applied to both the reversed flow and the connected components N of ∂M. It follows that all the leaves of the foliation are diffeomorphic to $\mathbb{R}$ as desired.

To verity (2) we see that ϕ has a source Ω_0 by the Spectral Theorem [19] (the proof is similar to the boundaryless case since M has non-empty interior). If $\Omega_0 \cap \partial M \neq \emptyset$, then $\Omega_0 \subset \partial M$ by Lemma 1(3) since Ω_0 is transitive and isolated. Since Ω_0 is a source we have $W^{ss}(z) \subset \Omega_0$ for all $z \in \Omega_0$. Then we would have $E^s_z \subset T_z(\partial M)$ for every $z \in \Omega_0$. This would imply $E^s_z \oplus E^u_z \subset T_z(\partial M)$ for all $z \in \Omega_0 \subset \partial M$ and so $T_z M = E^s_z \oplus E^s_z \oplus E^o_z \subset T_z(\partial M)$ a contradiction. The proof follows. $\square$

Proof of Theorem 1. Let M be a compact manifold with boundary. Suppose by contradiction that M supports a codimension one Anosov flow ϕ. If $dim(M) = 3$ we get a contradiction by Corollary 2 since any connected component of ∂M is an invariant closed surface of ϕ. If $dim(M) \geqslant 4$ we get a contradiction by Corollary 1 and Lemma 4. These contradiction prove the result. $\square$

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